The $d\delta$–lemma for weakly Lefschetz symplectic manifolds

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Abstract

For a symplectic manifold (M, ω), not necessarily hard Lefschetz, we prove a version of the Merkulov $d\delta$–lemma ([17, 4]). We also study the $d\delta$–lemma and related cohomologies for compact symplectic solvmanifolds.

1 Introduction

Let (M, ω) be a symplectic manifold, that is, M is a differentiable manifold of dimension $2n$ with a closed non-degenerate 2–form ω, the *symplectic form*. Denote by $\Omega^k(M)$ the space of the differential k–forms on M. According to Libermann [12] and Brylinski [2] there is a *symplectic star operator* $* : \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$ associated to the symplectic form ω satisfying $*^2 = Id$ (see Section 2 for the definition). Such an operator is the symplectic analogue of the Hodge star operator on oriented Riemannian manifolds. Then, one can define the *codifferential* $\delta = \pm * d *$ which satisfies $\delta^2 = 0$ and $d\delta + \delta d = 0$ (although δ does not satisfy a Leibniz rule [15]).

As in Riemannian Hodge theory, a k–form $\alpha \in \Omega^k(M)$ is said to be *coclosed* if $\delta\alpha = 0$, *coexact* if $\alpha = \delta\beta$ for some β, and *symplectically harmonic* if it is closed and coclosed. But, unlike the case of Riemannian manifolds, there are many symplectically harmonic forms which are exact. This is the reason for which for any $k \geq 0$, we define the space of *harmonic cohomology* $H^k_{\mathrm{hr}}(M, \omega)$ of degree k to be the subspace of the de Rham cohomology $H^k(M)$ consisting of all cohomology classes which contain at least one symplectically harmonic k–form.

Mathieu [16] and, independently, Yan [22] proved that $H^k_{\mathrm{hr}}(M, \omega) = H^k(M)$ for all k if and only if (M, ω) satisfies the hard Lefschetz property, i.e. the map

$$L^{n-k}\colon H^k(M) \longrightarrow H^{2n-k}(M)$$

given by $L^{n-k}[\alpha] = [\omega^{n-k} \wedge \alpha]$ is a surjection for all $k \leq n-1$. On the other hand, for *compact* symplectic manifolds, Merkulov and Cavalcanti ([17, 4]) showed that the existence of symplectic harmonic forms in every de Rham cohomology class is equivalent to the symplectic *$d\delta$–lemma*, that is, to the identities

$$\mathrm{Im}\, d \cap \ker \delta = \mathrm{Im}\, d\delta = \mathrm{Im}\, \delta \cap \ker d, \tag{1}$$

which mean that if α is a symplectically harmonic k–form and either is exact or coexact, then $\alpha = d\delta\beta$ for some k–form β.

Consider the subcomplex $(\Omega^*_\delta(M, \omega), d)$ of the de Rham complex $(\Omega^*(M), d)$ of M, where $\Omega^k_\delta(M, \omega)$ is the space of the coclosed k–forms. We denote by $H^*_\delta(M, \omega)$ its cohomology and by i the natural map

$$i\colon H^k_\delta(M, \omega) \longrightarrow H^k(M), \tag{2}$$

for all $k \geq 0$. In [8] Guillemin proved that if M is compact, then the map i is bijective if and only if (M, ω) is hard Lefschetz or, equivalently, it satisfies the $d\delta$–lemma.

In this paper, we aim to generalize these results to symplectic manifolds which are not hard Lefschetz. Recall the following definition from [6].

Definition 1.1 *A symplectic manifold (M, ω) of dimension $2n$ is said to be s–*Lefschetz*, where $0 \leq s \leq n-1$, if the map*

$$L^{n-k} \colon H^k(M) \longrightarrow H^{2n-k}(M)$$

is an epimorphism for all $k \leq s$. (If M is compact, then we actually have that L^{n-k} are isomorphisms because of Poincaré duality.)

Whenever (M, ω) is not hard Lefschetz, there is some integer number $s \geq 0$ such that (M, ω) is s–Lefschetz, but not $(s+1)$–Lefschetz. Note that (M, ω) is $(n-1)$–Lefschetz if it satisfies the hard Lefschetz theorem.

Concerning the harmonic cohomology for such manifolds, we have the following result.

Theorem 1.2 [7] *Let (M, ω) be a symplectic manifold of dimension $2n$ and let $s \leq n-1$. Then the following statements are equivalent:*

(i) *(M, ω) is s–Lefschetz.*

(ii) *$H^k_{\mathrm{hr}}(M, \omega) = H^k(M)$ for every $k \leq s+2$, and $H^{2n-k}_{\mathrm{hr}}(M, \omega) = H^{2n-k}(M)$ for every $k \leq s$.*

(iii) *$H^{2n-k}_{\mathrm{hr}}(M, \omega) = H^{2n-k}(M)$ for every $k \leq s$.*

Notice that Theorem 1.2 implies that every de Rham cohomology class of M admits a symplectically harmonic representative if and only if (M, ω) is hard Lefschetz, which is the result proved independently by Mathieu and Yan [16, 22].

For any non-hard Lefschetz symplectic manifold, it seems interesting to understand how the level s at which the Lefschetz property is lost affects to other properties of the manifold, such as the above mentioned $d\delta$–lemma, or to the properties of the map i. Our purpose in this paper is to explore these questions, as we explain below.

In Section 2 we recall some properties of the spaces of harmonic cohomology. In Section 3, we sharpen the result of Merkulov and the result of Guillemin by using the concept of s–Lefschetz property. We need first to weaken the condition of the $d\delta$–lemma to the following

Definition 1.3 *Let (M, ω) be a symplectic manifold of dimension $2n$, and $0 \leq s \leq n-1$. We say that (M, ω) satisfies the $d\delta$–*lemma up to degree *s if*

$$\begin{array}{lll} \text{(3)} & \operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta = \operatorname{Im} \delta \cap \ker d, & \text{on } \Omega^k(M), \text{ for } k \leq s, \\ & \operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta, & \text{on } \Omega^{s+1}(M). \end{array}$$

Therefore, if (M, ω) satisfies the $d\delta$–lemma up to degree s, and $\alpha \in \Omega^{\leq s}(M)$ is symplectically harmonic and either is exact or coexact then $\alpha = d\delta\beta$ for some β; moreover, if $\alpha \in \Omega^{s+1}(M)$ is symplectically harmonic and exact then $\alpha = d\delta\beta$ for some β.

Following the approach in Cavalcanti's proof [4] of the result of Merkulov we prove the following theorem.

Theorem 1.4 *($d\delta$–lemma for weakly Lefschetz manifolds). Let (M, ω) be a compact symplectic manifold of dimension $2n$ and let $s \leq n-1$. Then the following statements are equivalent:*

(i) (M,ω) *is s–Lefschetz.*

(ii) (M,ω) *satisfies the $d\delta$–lemma up to degree s.*

(iii) *The identities* (1) *hold on* $\Omega^{\geq(2n-s)}(M)$*, and* $\operatorname{Im}\delta\cap\ker d=\operatorname{Im}d\delta$ *holds on* $\Omega^{2n-s-1}(M)$*.*

In Section 3 we also show the following theorem regarding the map (2) for weakly symplectic manifolds.

Theorem 1.5 *Let (M,ω) be a compact symplectic manifold of dimension $2n$ and let $s\leq n-1$. Then the following statements are equivalent:*

(i) (M,ω) *is s–Lefschetz.*

(ii) *The map* $i\colon H^k_\delta(M,\omega)\longrightarrow H^k(M)$ *is bijective for all $k\leq s+1$ and for $k\geq 2n-s$.*

(iii) *The map* $i\colon H^k_\delta(M,\omega)\longrightarrow H^k(M)$ *is bijective for all $k\geq 2n-s$.*

The harmonic cohomology of compact symplectic nilmanifolds has been studied by different authors (see [22, 10, 21]). In Section 4 we consider compact solvmanifolds $M=\Gamma\backslash G$, where G is a simply connected solvable Lie group whose Lie algebra $\mathfrak{g}$ is completely solvable, i.e., the map $\mathrm{ad}_X\colon\mathfrak{g}\longrightarrow\mathfrak{g}$ has only real eigenvalues for any $X\in\mathfrak{g}$, and Γ is a discrete subgroup of G such that the quotient $M=\Gamma\backslash G$ is compact. We show that the harmonic cohomology of $(M=\Gamma\backslash G,\omega)$ is isomorphic to the harmonic cohomology at the level of the invariant forms. We exhibit some examples of compact symplectic solvmanifolds M which are s–Lefschetz but not $(s+1)$–Lefschetz, for small values of s, and so the map i is bijective for $k\geq 2n-s$ and they satisfy the $d\delta$–lemma up to degree s. We detect that they do not satisfy the $d\delta$–lemma up to degree $s+1$ by exhibiting an invariant symplectically harmonic $(s+1)$–form x such that $x\in\operatorname{Im}\delta$ but $x\notin\operatorname{Im}d$. We also find an invariant class $u\in H^{2n-s-1}_\delta(M,\omega)$ such that $i(u)=0$ in $H^{2n-s-1}(M)$.

2 Harmonic cohomology of s–Lefschetz manifolds

We recall some definitions and results about the spaces of harmonic cohomology classes that we will need in the following sections. Let (M,ω) be a symplectic manifold of dimension $2n$. Denote by $\Omega^*(M)$ the algebra of differential forms on M, by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M, and by $\mathcal{F}(M)$ the algebra of differentiable functions on M. Since ω is a non-degenerate 2–form, we have the volume form $v_M=\frac{\omega^n}{n!}$, and the isomorphism

$$\natural:\mathfrak{X}(M)\longrightarrow\Omega^1(M)$$

defined by $\natural(X)=\iota_X(\omega)$ for $X\in\mathfrak{X}(M)$, where ι_X denotes the contraction by X. We extend $\natural$ to an isomorphism of graded algebras $\natural:\bigoplus_{k\geq 0}\mathfrak{X}^k(M)\longrightarrow\bigoplus_{k\geq 0}\Omega^k(M)$, where $\mathfrak{X}^k(M)$ denotes the space of the skew-symmetric k–vectors fields. Libermann (see [12, 13]) defined the *symplectic star operator*

$$*:\Omega^k(M)\longrightarrow\Omega^{2n-k}(M)$$

by the condition

$$*(\alpha)=(-1)^k\iota_{\natural^{-1}(\alpha)}(v_M).$$

This operator can be also defined in terms of the skew-symmetric bivector field G dual to ω, that is, $G=-\natural^{-1}(\omega)$. (G is the unique non-degenerate Poisson structure [14] associated with

ω.) Denote by $\Lambda^k(G)$, $k \geq 0$, the associated pairing $\Lambda^k(G) : \Omega^k(M) \times \Omega^k(M) \longrightarrow \mathcal{F}(M)$ which is $(-1)^k$–symmetric (i.e. symmetric for even k, anti-symmetric for odd k). Imitating the Hodge star operator for oriented Riemannian manifolds, Brylinski [2] defined the *symplectic star operator* by the condition $\beta \wedge (*\alpha) = \Lambda^k(G)(\beta, \alpha)v_M$, for $\alpha, \beta \in \Omega^k(M)$. An easy consequence is that $*^2 = \mathrm{Id}$.

Koszul [11] introduced the differential $\delta \colon \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ on any Poisson manifold M, with Poisson tensor G, by the condition

$$\delta = [\iota_G, d],$$

and he proved that $\delta^2 = d\delta + \delta d = 0$. Later work by Brylinski [2], shows that the Koszul differential is a *symplectic codifferential* of the exterior differential with respect to the *symplectic star operator*, that is,

$$\delta\alpha = (-1)^{k+1} * d * (\alpha),$$

for $\alpha \in \Omega^k(M)$. As in Riemannian Hodge theory, a k–form $\alpha \in \Omega^k(M)$ is said to be *coclosed* if $\delta\alpha = 0$, *coexact* if $\alpha = \delta\beta$ for some β, and *symplectically harmonic* if it is closed and coclosed. Notice that Koszul definition of δ implies that if α is closed, then $\delta\alpha$ is exact. In [15] it is proved the following Leibniz rule for δ. If f is an arbitrary differentiable function on M and $\alpha \in \Omega^*(M)$, then

$$\delta(f\alpha) = f\,\delta\alpha - \iota_{X_f}(\alpha),$$

where X_f is the Hamiltonian vector field of f, i.e., $\iota_{X_f}(\omega) = df$.

Let $\Omega^k_{\mathrm{hr}}(M, \omega) = \{\alpha \in \Omega^k(M) \mid d\alpha = \delta\alpha = 0\}$ be the space of the symplectically harmonic k–forms. For the de Rham cohomology classes of M, we consider the vector space

$$H^k_{\mathrm{hr}}(M, \omega) = \frac{\Omega^k_{\mathrm{hr}}(M, \omega)}{\Omega^k_{\mathrm{hr}}(M, \omega) \cap \mathrm{Im}\, d},$$

consisting of the cohomology classes in $H^k(M)$ containing at least one symplectically harmonic form.

For $p, k \geq 0$ we define

$$L^p : \Omega^k(M) \longrightarrow \Omega^{2p+k}(M)$$

by $L^p(\alpha) = \omega^p \wedge \alpha$ for $\alpha \in \Omega^k(M)$. In [22] it is proved the property following

Lemma 2.1 [22] *(Duality on differential forms). The map*

$$L^{n-k} \colon \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$$

is an isomorphism for $0 \leq k \leq n-1$.

Since ω^p is closed, we have

$$[L^p, d] = L^p \circ d - d \circ L^p = 0,$$

and the map L^p induces a map $L^p : H^k(M) \longrightarrow H^{2p+k}(M)$ on cohomology. However, the isomorphisms of Lemma 2.1 do not imply special properties on the maps on cohomology (see Definition 1.1). Relations between the operators ι_G, L, d and δ were proved by Yan in [22]. Here we mention the following

$$\iota_G = - * L *, \quad [\iota_G, \delta] = 0, \quad [L, \delta] = -d,$$

which implies that if α is coclosed then $d\alpha$ is coexact, and if α is a symplectically harmonic form then $L\alpha$ and $\iota_G\alpha$ are symplectically harmonic. Also in [22] the following is proved.

Lemma 2.2 [22] *(Duality on harmonic forms). The map*

$$L^{n-k}\colon \Omega^k_{\rm hr}(M,\omega) \longrightarrow \Omega^{2n-k}_{\rm hr}(M,\omega)$$

is an isomorphism for $0 \le k \le n-1$.

Lemma 2.2 implies that the homomorphism

$$L^{n-k}\colon H^k_{\rm hr}(M,\omega) \longrightarrow H^{2n-k}_{\rm hr}(M,\omega)$$

is surjective. (Notice that the duality on harmonic forms may be not satisfied at the level of the spaces $H^*_{\rm hr}(M,\omega)$.) Since $H^{2n-k}_{\rm hr}(M,\omega)$ is a subspace of the de Rham cohomology $H^{2n-k}(M)$, we conclude that (see [10, Corollary 1.7])

$$H^{2n-k}_{\rm hr}(M,\omega) = \mathrm{Im}\,(L^{n-k}\colon H^k_{\rm hr}(M,\omega) \longrightarrow H^{2n-k}(M)).$$

A nonzero k–form α, with $k \le n$, is called *primitive* (or *effective*) if $L^{n-k+1}(\alpha) = 0$. Thus, any 1–form is primitive.

Lemma 2.3 [13, page 46] *If α is a primitive k–form, then there is a constant c such that its symplectic star operator $*\alpha$ satisfies $*\alpha = c\,L^{n-k}(\alpha)$.*

Notice that the previous lemma implies that every closed primitive k–form is symplectically harmonic, and in particular $H^1(M) = H^1_{\rm hr}(M,\omega)$. For the classes in $H^2(M)$, Mathieu proved that any cohomology class of degree 2 has a symplectically harmonic representative, i.e., $H^2(M) = H^2_{\rm hr}(M,\omega)$.

Lemma 2.4 *Let α a k–form with $k \le n$. Then, α is primitive if and only if $\iota_G(\alpha) = 0$.*

Proof : It follows from the identity $\iota_G = - * L *$ and Lemma 2.3. If α is primitive, $\iota_G(\alpha) = - * L * (\alpha) = - * c\,L^{n-k+1}(\alpha) = 0$. $\boxed{QED}$

Lemma 2.5 *If α is a primitive k–form then, for all $j \le n-k$, there is a non-zero constant $c_{j,k}$ such that $\iota_G^j L^j(\alpha) = c_{j,k}\,\alpha$.*

Proof : In [22] it is proved the relation $[\iota_G, L] = A$, where $A = \sum (n-k)\pi_k$, π_k being the projection. Thus, for $j = 1$ we have that $\iota_G L(\alpha) = A\alpha + L(\iota_G\alpha) = (n-k)\alpha$ because α is primitive. Suppose that $\iota_G^j L^j(\alpha) = c_{j,k}\,\alpha$ for some $j < n-k$ with $c_{j,k}$ a non-zero constant. Hence, $\iota_G^{j+1}L^{j+1}(\alpha) = \iota_G^j \iota_G L L^j(\alpha) = \iota_G^j L \iota_G L^j(\alpha) + (n-k-2j)\iota_G^j L^j(\alpha) = \iota_G^j L \iota_G L^j(\alpha) + (n-k-2j)c_{j,k}\,(\alpha)$ by the induction hypothesis. After p times we get that

$$\iota_G^{j+1}L^{j+1}(\alpha) = \iota_G^{j-p} L \iota_G^{p+1} L^j(\alpha) + (p+1)(n-k-2j+p)c_{j,k}\,\alpha.$$

Therefore, for $p = j-1$ and using the induction hypothesis we conclude that $\iota_G^{j+1}L^{j+1}(\alpha) = c_{j+1,k}\,\alpha$, with $c_{j+1,k} = (j+1)(n-k-j)c_{j,k}$ a non-zero constant. $\boxed{QED}$

3 The $d\delta$–lemma for s–Lefschetz manifolds

This section is devoted to the study of the $d\delta$–lemma for symplectic manifolds which are not necessarily hard Lefschetz. By Definition 1.3, (M,ω) satisfies the $d\delta$-lemma up to degree s if $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta = \operatorname{Im} \delta \cap \ker d$ on $\Omega^k(M)$, for $k \leq s$ and $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ on $\Omega^{s+1}(M)$. By applying duality with the symplectic $*$-operator, this is equivalent to

$$
(4) \qquad \begin{array}{ll} \operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta = \operatorname{Im} d \cap \ker \delta, & \text{on } \Omega^{2n-k}(M), \text{ for } k \leq s, \\ \operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta, & \text{on } \Omega^{2n-s-1}(M). \end{array}
$$

Let us see the one implication (the other one is proved in an analogous way). Suppose that (M,ω) satisfies the $d\delta$-lemma up to degree s. If $\alpha_{2n-k} \in \Omega^{2n-k}(M)$, $0 \leq k \leq s+1$, satisfies that $\alpha_{2n-k} \in \operatorname{Im} \delta \cap \ker d$, then $*\alpha_{2n-k}$ is a k–form in $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$, so there is a k–form β_k such that $*\alpha_{2n-k} = d\delta(\beta_k)$ and hence $\alpha_{2n-k} = *d\delta(\beta_k) = -\delta d(*\beta_k) = d\delta(*\beta_k)$. The equality $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ on $\Omega^{\geq(2n-s)}(M)$ is proved analogously.

Note that if (M,ω) satisfies the $d\delta$–lemma up to degree $n-1$ then both (3) and (4) hold for $s = n-1$, and hence (M,ω) satisfies the $d\delta$–lemma since then (1) also holds on the space $\Omega^n(M)$.

In order to prove Theorems 1.4 and 1.5 we need the following results.

Lemma 3.1 *Let (M,ω) be a symplectic manifold of dimension $2n$, and let α be a k–form. Then*

(i) $d\delta(L^p(\alpha)) = L^p(d\delta(\alpha))$ *for all* $p \geq 0$.

(ii) *If α is primitive, then $d\delta(\alpha)$ is also primitive.*

Proof : Since $[L,\delta] = -d$, we see that $\delta L = L\delta + d$. Thus, $d\delta(L^p(\alpha)) = d(L\delta + d)L^{p-1}(\alpha) = dL\delta L^{p-1}(\alpha)$. Proceeding in this fashion p times, and using that L and d commute, we obtain (i). Now to show (ii) we have, using (i), that $L^{n-k+1}(d\delta(\alpha)) = d\delta(L^{n-k+1}(\alpha)) = 0$ since α is a primitive k–form. $\boxed{QED}$

Lemma 3.2 *Let (M,ω) be a symplectic manifold of dimension $2n$, let β be a r–form, and let $\alpha = L^p(\beta)$, with $p \geq 0$. If $\delta\beta$ is exact, then $\delta\alpha$ is also exact.*

Proof : Write $\delta\beta = d\gamma$. Using $[L,\delta] = -d$, we have $\delta\alpha = \delta L^p(\beta) = (L\delta + d)L^{p-1}(\beta) = L\delta L^{p-1}(\beta) + dL^{p-1}(\beta)$. Proceeding in a similar way with the first summand, after p steps, we get

$$\delta\alpha = \delta L^p(\beta) = L^p\delta(\beta) + p\, dL^{p-1}(\beta) = d(L^p(\gamma) + pL^{p-1}(\beta)),$$

which proves the lemma. $\boxed{QED}$

Consider a $(2n-i)$–form ψ on (M,ω) with $i \leq n$. According to the duality on differential forms, there is a unique i–form φ such that $\psi = L^{n-i}(\varphi)$. Lepage decomposition theorem [13] implies that φ may be uniquely decomposed as a sum

$$(5) \qquad \varphi = \varphi_i + L(\varphi_{i-2}) + \cdots + L^q(\varphi_{i-2q}),$$

with $q \leq [i/2]$, where $[i/2]$ being the largest integer less than or equal to $i/2$, and where the form φ_{i-2j} is a primitive $(i-2j)$–form, for $j = 0, \ldots, q$. This implies that $\psi = L^{n-i}(\varphi)$ may be uniquely decomposed as the sum

$$(6) \qquad \psi = L^{n-i}(\varphi) = L^{n-i}(\varphi_i) + L^{n-i+1}(\varphi_{i-2}) + \cdots + L^{n-i+q}(\varphi_{i-2q}).$$

Lemma 3.3 *Let (M,ω) be a symplectic manifold of dimension $2n$, and let $\psi = L^{n-i}(\varphi) \in \Omega^{2n-i}(M)$ with $i \leq n$.*

(i) *If $d\delta(\psi) = 0$, or equivalently $d\delta(\varphi) = 0$, then all the forms φ_{i-2j} in the decomposition* (5) *and* (6) *satisfy $d\delta(\varphi_{i-2j}) = 0$.*

(ii) *If $\delta\varphi_{i-2j}$ is exact for all $j = 0, \ldots, q$, then both $\delta\varphi$ and $\delta\psi$ are exact.*

Proof : Suppose that $d\delta(\psi) = 0$. Applying $d\delta$ to (6), using Lemma 3.1 and the uniqueness of the decomposition, we have that

$$L^{n-i+j} d\delta(\varphi_{i-2j}) = 0,$$

for $j = 0, \ldots, q$. We see that $L^{n-i+j} d\delta(\varphi_{i-2j}) = 0$ implies $d\delta(\varphi_{i-2j}) = 0$. In fact, the map $L^{n-i+2j} \colon \Omega^{i-2j}(M) \longrightarrow \Omega^{2n-i+2j}(M)$ is an isomorphism for all $j = 0, \ldots, q$. So, the map $L^{n-i+j} \colon \Omega^{i-2j}(M) \longrightarrow \Omega^{2n-i}(M)$ is injective for $j = 1, \ldots, q$, and it is an isomorphism for $j = 0$. Using again Lemma 3.1 and the duality on differential forms, one can check that $d\delta(\varphi) = 0$ implies the same result. Part (ii) follows from Lemma 3.2 and using that δ is a linear map. $\boxed{QED}$

Proposition 3.4 *Let (M,ω) be an s–Lefschetz compact symplectic manifold of dimension $2n$. Then,*

$$\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \operatorname{Im}\delta,$$

on the spaces $\Omega^{\leq s}(M)$ and $\Omega^{\geq (2n-s-2)}(M)$; and

$$\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d \cap \operatorname{Im}\delta,$$

on $\Omega^{\leq (s+2)}(M)$ and $\Omega^{\geq (2n-s)}(M)$.

Proof : We prove only the first identity because the second is analogous by duality using the symplectic $*$-operator. The result can be restated in the following way: if φ is a k–form, with $k \leq s+1$ or $k \geq 2n-s-1$, and such that $d\delta(\varphi) = 0$, then $\delta\varphi$ is exact.

First, we show such a result for any primitive k–form φ with $k \leq s+1$. We define the $(k-1)$–form γ by

$$L^{n-k+1}(\gamma) = dL^{n-k}(\varphi). \tag{7}$$

Thus γ is primitive since $L^{n-k+2}(\gamma) = dL^{n-k+1}(\varphi) = 0$. Applying ι_G^{n-k+1} in (7), using Lemma 2.5 and $\delta = [\iota_G, d]$, we have

$$c_{n-k+1,k-1}\gamma = \iota_G^{n-k+1} dL^{n-k}\varphi = \iota_G^{n-k}(d\iota_G + \delta)L^{n-k}\varphi.$$

Proceeding in this fashion, after $(n-k+1)$ times, we have

$$c_{n-k+1,k-1}\gamma = (d\iota_G^{n-k+1} - (n-k+1)\delta\iota_G^{n-k})L^{n-k}\varphi.$$

Since φ is primitive, $(d\iota_G^{n-k+1})L^{n-k}\varphi = d\iota_G(c_{n-k,k}\varphi) = 0$ by Lemma 2.4. So, there is a non-zero constant c such that $\gamma = c\,\delta\varphi$. Applying L^{n-k+1} to both sides and using (7) we obtain

$$c\,L^{n-k+1}\delta\varphi = L^{n-k+1}\gamma = d(L^{n-k}\varphi).$$

By hypothesis $\delta\varphi$ is closed. Moreover the map $L^{n-k+1} \colon H^{k-1}(M) \longrightarrow H^{2n-k+1}(M)$ is an isomorphism for $k-1 \leq s$ since (M,ω) is compact and s–Lefschetz. Thus $\delta\varphi$ is exact because $L^{n-k+1}\delta\varphi$ defines the zero class.

Now we pass to the case where φ is an arbitrary k–form with $k \leq s+1$ such that $d\delta(\varphi) = 0$. From Lemma 3.3 we know that every primitive form φ_{i-2j} in the decomposition (5) satisfies $d\delta(\varphi_{i-2j}) = 0$, and so $\delta(\varphi_{i-2j})$ is exact. Now Lemma 3.3 implies that $\delta\varphi$ is exact.

Finally, if ψ is a k–form with $k \geq 2n-s-1$ and such that $d\delta(\psi) = 0$, then the forms φ_{i-2j} in the decomposition (6) are of degree $\leq s+1$, and they satisfy $d\delta(\varphi_{i-2j}) = 0$ by Lemma 3.3. Taking account the previous result for primitive forms, we conclude that all the forms $\delta\varphi_{i-2j}$ are exact, and hence $\delta\psi$ is exact by Lemma 3.3. $\boxed{QED}$

Proposition 3.5 *Let (M,ω) be an s–Lefschetz compact symplectic manifold of dimension $2n$. We have*

(i) $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \ker\delta$ *on* $\Omega^{\leq s}(M)$ *and* $\Omega^{\geq(2n-s)}(M)$.

(ii) (M,ω) *satisfies the $d\delta$–lemma up to degree s.*

Proof : Part (i) follows directly from Proposition 3.4.

To show (ii), we shall first prove that $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on the spaces $\Omega^{\geq(2n-s-1)}(M)$. We will prove this by induction on s. For $s=0$, assume $\alpha \in \Omega^{2n}(M)$ such that $\alpha \in \operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \ker\delta$. Then $\alpha = 0$ because $\operatorname{Im}\delta = 0$, and so $\alpha = 0 = d\delta 0$. Now we see that $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on $\Omega^{2n-1}(M)$. Let $\beta = \delta\alpha$ be a $(2n-1)$–form, $\alpha \in \Omega^{2n}(M)$, such that $d\delta\alpha = 0$. Since $d\alpha = 0$ and $H^{2n}(M) = H^{2n}_{\mathrm{hr}}(M,\omega)$, there is $\tilde{\alpha} \in \Omega^{2n}(M)$ such that $\delta\tilde{\alpha} = 0$, $d\tilde{\alpha} = 0$ and $\alpha = \tilde{\alpha} + d\gamma$, for some $\gamma \in \Omega^{2n-1}(M)$. Therefore, $\beta = \delta\alpha = \delta d\gamma = d\delta(-\gamma)$.

Now take $s > 0$, and assume that if (M,ω) is $(s-1)$–Lefschetz, then $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on $\Omega^{\geq(2n-s)}(M)$. We need to prove that if (M,ω) is s–Lefschetz, then $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on $\Omega^{2n-s-1}(M)$. We will use subscripts to keep track of the spaces that the forms belong to, i.e. $\alpha_k \in \Omega^k(M)$. We consider a $(2n-s-1)$–form α_{2n-s-1} such that $\alpha_{2n-s-1} = \delta\alpha_{2n-s} \in \operatorname{Im}\delta \cap \ker d$. Then,

$$0 = d\alpha_{2n-s-1} = d\delta\alpha_{2n-s} = -\delta d\alpha_{2n-s},$$

which implies that $d\alpha_{2n-s}$ is a $(2n-s+1)$–form such that $d\alpha_{2n-s} \in \operatorname{Im} d \cap \ker\delta = \operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ by (i) and induction hypothesis. Thus

$$d\alpha_{2n-s} = d\delta\alpha_{2n-s+1},$$

for some $\alpha_{2n-s+1} \in \Omega^{2n-s+1}(M)$, and consequently

$$d(\alpha_{2n-s} - \delta\alpha_{2n-s+1}) = 0,$$

which means that $(\alpha_{2n-s} - \delta\alpha_{2n-s+1})$ defines a de Rham cohomology class in $H^{2n-s}(M) = H^{2n-s}_{\mathrm{hr}}(M,\omega)$, the last equality by Theorem 1.2. Thus, there exist a symplectically harmonic $(2n-s)$–form β_{2n-s} and $\eta_{2n-s-1} \in \Omega^{2n-s-1}(M)$ such that

$$\alpha_{2n-s} - \delta\alpha_{2n-s+1} - \beta_{2n-s} = d\eta_{2n-s-1}.$$

Applying δ to both sides we have

$$\alpha_{2n-s-1} = \delta\alpha_{2n-2} = \delta d\eta_{2n-s-1} = -d\delta\eta_{2n-s-1} \in \operatorname{Im} d\delta.$$

To end the proof, we use the duality by the symplectic $*$-operator to show that $\operatorname{Im} d \cap \ker\delta = \operatorname{Im} d\delta$ on the spaces $\Omega^{\leq(s+1)}(M)$. In fact, let us consider α_r a differential r–form, with $r \leq s+1$, such that $\alpha_r \in \operatorname{Im} d \cap \ker\delta$. Then, $*\alpha_r$ is a $(2n-r)$–form, $2n-r \geq 2n-s-1$, such that

$*\alpha_r \in \operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$, and so $\alpha_r \in \operatorname{Im} d\delta$. The equality $\operatorname{Im} d \cap \ker\delta = \operatorname{Im}\delta \cap \ker d$ on the spaces $\Omega^{\leq s}(M)$ follows from (i), and this completes the proof of the $d\delta$-lemma up to degree s for (M, ω). $\boxed{QED}$

Proof of Theorem 1.4 : Clearly (i) implies (ii) by Proposition 3.5. Also (ii) implies (iii) by duality of the symplectic $*$-operator.

Let us show that (iii) implies (i). By Theorem 1.2, it is enough to prove that every de Rham cohomology class of degree k has a symplectically harmonic representative for $2n-s \leq k \leq 2n$. Let us consider $[\gamma] \in H^k(M)$ with $2n-s \leq k \leq 2n$. Then $d\gamma = 0$, and $\delta\gamma$ is a $(k-1)$–form such that $d\delta\gamma = 0$ since d and δ anticommute. This means that $\delta\gamma$ lives in $\operatorname{Im}\delta \cap \ker d$ which is equal to $\operatorname{Im} d\delta$ on forms of degree $k-1 \geq 2n-s-1$ by the hypothesis (iii). This implies that there is a $(k-1)$–form θ such that $\delta\gamma = d\delta\theta$. So $\delta(\gamma + d\theta) = 0$. Then, the form $\gamma + d\theta$ is symplectically harmonic and cohomologous to γ. $\boxed{QED}$

Remark 3.6 *Notice that if (M, ω) is a compact symplectic manifold of dimension $2n$ and it is $(n-2)$–Lefschetz, then the identities* (1) *hold on $\Omega^{\leq(n-2)}(M)$ and $\Omega^{\geq(n+2)}(M)$, and also $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \operatorname{Im}\delta = \operatorname{Im} d \cap \ker\delta$ on $\Omega^n(M)$, by Proposition 3.4. Nonetheless, if (M, ω) is not hard Lefschetz, then this last space is in general different from* $\operatorname{Im} d\delta$.

Let $\Omega^k_\delta(M, \omega) = \{\alpha \in \Omega^k(M) \mid \delta\alpha = 0\}$ be the space of the coclosed k–forms. Since d and δ anti-commute, then $d(\Omega^k_\delta(M, \omega)) \subset \Omega^{k+1}_\delta(M, \omega)$, and so $(\Omega^*_\delta(M, \omega), d)$ is a subcomplex of the de Rham complex $(\Omega^*(M), d)$. We denote by $H^*_\delta(M, \omega)$ its cohomology, that is

$$H^k_\delta(M, \omega) = \frac{\ker(d\colon \Omega^k_\delta(M, \omega) \longrightarrow \Omega^{k+1}_\delta(M, \omega))}{\operatorname{Im}(d\colon \Omega^{k-1}_\delta(M, \omega) \longrightarrow \Omega^k_\delta(M, \omega))}.$$

Therefore, any cohomology class on $H^k_\delta(M, \omega)$ is symplectically harmonic, and we have a natural map $i_1\colon H^k_\delta(M, \omega) \longrightarrow H^k_{\mathrm{hr}}(M, \omega)$ which is always surjective but may be non-injective. The next theorem gives a necessary and sufficient condition for the injectivity of the map i_1. (Notice that $\Omega^*_\delta(M, \omega) = \bigoplus \Omega^k_\delta(M, \omega)$ is a vector space but not an algebra because the codifferential δ does not satisfy a Leibniz rule.) It is clear that there is a natural map

$$i\colon H^k_\delta(M, \omega) \longrightarrow H^k(M),$$

for all k. In fact, denote by i_2 the natural inclusion

$$i_2\colon H^k_{\mathrm{hr}}(M, \omega) \longrightarrow H^k(M).$$

Then, $i = i_2 \circ i_1$.

Proof of Theorem 1.5 : Suppose that (M, ω) is s–Lefschetz. By Theorem 1.2, $H^k_{\mathrm{hr}}(M, \omega) = H^k(M)$ for $k \leq s+2$ and $k \geq 2n-s$. Then, to show (ii) it is enough to prove that the map $i = i_1\colon H^k_\delta(M, \omega) \longrightarrow H^k_{\mathrm{hr}}(M, \omega)$ is injective for $k \leq s+1$ and $k \geq 2n-s$ because such a map is always surjective. Consider $[\alpha] \in H^k_\delta(M, \omega)$ and suppose that $[\alpha] = i[\alpha]$ defines the zero class on $H^k_{\mathrm{hr}}(M, \omega)$. Then α is exact, i.e. $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$. But if $k \leq s+1$ or $k \geq 2n-s$, Theorem 1.4 implies $\alpha = d\delta\eta$ for some $\eta \in \Omega^k(M)$. Hence $\alpha = d\delta\eta \in \operatorname{Im}(d\colon \Omega^{k-1}_\delta(M, \omega) \longrightarrow \Omega^k_\delta(M, \omega))$. This means that α defines the zero class on $H^k_\delta(M, \omega)$, which proves (ii).

Clearly (ii) implies (iii). We show that (iii) implies (i). In fact, if $[\alpha] \in H^k_\delta(M, \omega)$, $[\alpha]$ is a harmonic cohomology class. Thus, if the map $i\colon H^k_\delta(M, \omega) \longrightarrow H^k(M)$ is bijective for $k \geq 2n-s$ then $H^k_{\mathrm{hr}}(M, \omega) = H^k(M)$ for $k \geq 2n-s$, i.e. (M, ω) is s–Lefschetz according to Theorem 1.2. $\boxed{QED}$

4 Harmonic cohomology of compact completely solvmanifolds

Let $\mathfrak{g}$ be a Lie algebra of dimension $2n$, and denote by d the Chevalley-Eilenberg differential of $\mathfrak{g}$. An element $\omega \in \bigwedge^2(\mathfrak{g}^*)$ such that $d\omega = 0$ and $\omega^n \neq 0$ will be called a *symplectic form* on $\mathfrak{g}$.

Symplectic Hodge theory can be introduced for a symplectic form ω on a Lie algebra $\mathfrak{g}$ in a similar way as in Section 2. Let us define the star operator $*\colon \bigwedge^k(\mathfrak{g}^*) \longrightarrow \bigwedge^{2n-k}(\mathfrak{g}^*)$ by

$$* \alpha = (-1)^k \iota_{\natural^{-1}(\alpha)} \frac{\omega^n}{n!},$$

for any $\alpha \in \bigwedge^k(\mathfrak{g}^*)$, where $\natural$ denotes the isomorphism between $\bigwedge^k(\mathfrak{g})$ and $\bigwedge^k(\mathfrak{g}^*)$ extended from the natural isomorphism $\natural\colon \mathfrak{g} \longrightarrow \mathfrak{g}^*$ given by $\natural(X)(Y) = \omega(X, Y)$, for $X, Y \in \mathfrak{g}$.

We define the codifferential $\delta\colon \bigwedge^k(\mathfrak{g}^*) \longrightarrow \bigwedge^{k-1}(\mathfrak{g}^*)$ by

$$\delta\alpha = (-1)^{k+1} * d * \alpha,$$

for any $\alpha \in \bigwedge^k(\mathfrak{g}^*)$. Now, let $\bigwedge^k_{\mathrm{hr}}(\mathfrak{g}^*, \omega) = \{\alpha \in \bigwedge^k(\mathfrak{g}^*) \mid d\alpha = \delta\alpha = 0\}$, and consider the space

$$H^k_{\mathrm{hr}}(\mathfrak{g}, \omega) = \frac{\bigwedge^k_{\mathrm{hr}}(\mathfrak{g}^*, \omega)}{\bigwedge^k_{\mathrm{hr}}(\mathfrak{g}^*, \omega) \cap \operatorname{Im} d}.$$

Then, $H^k_{\mathrm{hr}}(\mathfrak{g}, \omega)$ consists of all the classes in the Chevalley-Eilenberg cohomology $H^k(\mathfrak{g})$ of $\mathfrak{g}$ containing at least one representative which is both closed and ω-coclosed.

Let $G \in \bigwedge^2(\mathfrak{g})$ be given by $G = -\natural^{-1}(\omega)$. In order to study the spaces $H^k_{\mathrm{hr}}(\mathfrak{g}, \omega)$ we consider the linear maps $L\colon \bigwedge^*(\mathfrak{g}^*) \longrightarrow \bigwedge^{*+2}(\mathfrak{g}^*)$, $\iota_G\colon \bigwedge^*(\mathfrak{g}^*) \longrightarrow \bigwedge^{*-2}(\mathfrak{g}^*)$ and $A\colon \bigwedge^*(\mathfrak{g}^*) \longrightarrow \bigwedge^*(\mathfrak{g}^*)$ as usual: $L\alpha$ is the wedge product by ω, $\iota_G\alpha$ the contraction by G and $A = \sum(n-k)\pi_k$, where π_k is the projection onto $\bigwedge^k(\mathfrak{g}^*)$. Following [22], although the arguments in this special case are more direct, the following relations hold:

$$[L, \delta] = -d, \quad [\iota_G, d] = \delta, \quad [L, d] = [\iota_G, \delta] = 0,$$

and

$$[\iota_G, L] = A, \quad [A, \iota_G] = 2\,\iota_G, \quad [A, L] = -2\,L.$$

Since the standard basis $\left\{ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ of $\mathfrak{sl}(2, \mathbb{C})$ satisfies

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y,$$

we have representations $\rho_1\colon \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathfrak{gl}(\bigwedge^*(\mathfrak{g}^*) \otimes \mathbb{C})$ and $\rho_2\colon \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathfrak{gl}(\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*, \omega) \otimes \mathbb{C})$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ on the complex vector spaces $\bigwedge^*(\mathfrak{g}^*) \otimes \mathbb{C}$ and $\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*, \omega) \otimes \mathbb{C}$, respectively, defined by the correspondence

$$\rho_i(X) = \iota_G, \quad \rho_i(Y) = L, \quad \rho_i(H) = A \quad (i = 1, 2),$$

where ι_G, L and A are understood for ρ_1 as the extension of the maps ι_G, L and A above to the complexification $\bigwedge^*(\mathfrak{g}^*) \otimes \mathbb{C}$ of $\bigwedge^*(\mathfrak{g}^*)$, and for ρ_2 as the restriction of them to the subspace $\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*, \omega) \otimes \mathbb{C}$. Notice that we can consider the restriction ρ_2 of the $\mathfrak{sl}(2, \mathbb{C})$ representation ρ_1 since if α is symplectically harmonic then $L\alpha$ and $\iota_G\alpha$ are symplectically harmonic.

It is well-known (see for example [20]) that for any representation ρ of $\mathfrak{sl}(2, \mathbb{C})$ on a finite dimensional complex vector space V, all the eigenvalues of $\rho(H)\colon V \longrightarrow V$ are integer numbers and, if V_k denotes the eigenspace of $\rho(H)$ with respect to the eigenvalue k, then

$$\rho(Y)^k\colon V_{-k} \longrightarrow V_k \quad \text{and} \quad \rho(X)^k\colon V_k \longrightarrow V_{-k}$$

are isomorphisms. Therefore, since $\bigwedge^r(\mathfrak{g}^*) \otimes \mathbb{C}$ and $\bigwedge^r_{\text{hr}}(\mathfrak{g}^*, \omega) \otimes \mathbb{C}$ are the eigenspaces of $\rho_1(H)$ and $\rho_2(H)$, respectively, with respect to the eigenvalue r, we conclude that

$$L^k\colon \bigwedge^{n-k}(\mathfrak{g}^*) \longrightarrow \bigwedge^{n+k}(\mathfrak{g}^*)$$

and

$$L^k\colon \bigwedge^{n-k}_{\text{hr}}(\mathfrak{g}^*, \omega) \longrightarrow \bigwedge^{n+k}_{\text{hr}}(\mathfrak{g}^*, \omega)$$

are isomorphisms for $k \geq 0$.

Remark 4.1 *Lemmas 2.1 and 2.2 expressing duality of forms and of harmonic forms, respectively, are derived by Yan [22] from the theory of a special type of infinite dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ called of finite H-spectrum. Any finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is of this type.*

The following result is a direct consequence of the isomorphisms L^k given above. In the proof we follow the lines of [21, Lemma 4.3] and [10, Corollary 2.4], where a similar result is given for the harmonic cohomology $H^*_{\text{hr}}(M)$ of a symplectic manifold M.

Lemma 4.2 *Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra with a symplectic form ω. For every $k \geq 0$, we have*

$$H^{n-k}_{\text{hr}}(\mathfrak{g}, \omega) = P_{n-k}(\mathfrak{g}, \omega) + L(H^{n-k-2}_{\text{hr}}(\mathfrak{g}, \omega)) \quad \text{and} \quad H^{n+k}_{\text{hr}}(\mathfrak{g}, \omega) = L^k(H^{n-k}_{\text{hr}}(\mathfrak{g}, \omega)),$$

where $P_r(\mathfrak{g}, \omega) = \{[\alpha] \in H^r(\mathfrak{g}) \mid L^{n-r+1}[\alpha] = 0\}$ is the space of primitive cohomology classes of degree r, and L denotes the product by $[\omega] \in H^2(\mathfrak{g})$.

Proof : Let $[\alpha] \in H^{n-k}_{\text{hr}}(\mathfrak{g}, \omega)$. Since $L^{k+2}(\bigwedge^{n-k-2}_{\text{hr}}(\mathfrak{g}^*, \omega)) = \bigwedge^{n+k+2}_{\text{hr}}(\mathfrak{g}^*, \omega)$, there exists β such that $d\beta = \delta\beta = 0$ and $L^{k+2}\beta = L^{k+1}\alpha$. Therefore, $L^{k+1}([\alpha] - L[\beta]) = 0$. Since $[\alpha] = ([\alpha] - L[\beta]) + L[\beta]$, the inclusion $H^{n-k}_{\text{hr}}(\mathfrak{g}, \omega) \subset P_{n-k}(\mathfrak{g}, \omega) + L(H^{n-k-2}_{\text{hr}}(\mathfrak{g}, \omega))$ holds.

To prove the other inclusion it suffices to show that any class $[\alpha] \in P_{n-k}(\mathfrak{g}, \omega)$ contains a representative $\tilde{\alpha}$ such that $\delta\tilde{\alpha} = 0$. Since $L^{k+1}[\alpha] = 0$, there exists $\gamma \in \bigwedge^{n+k+1}(\mathfrak{g}^*)$ such that $L^{k+1}\alpha = d\gamma$, so $\gamma = L^{k+1}\beta$ for some $\beta \in \bigwedge^{n-k-1}(\mathfrak{g}^*)$. Let $\tilde{\alpha} = \alpha - d\beta$. Since $L^{k+1}\tilde{\alpha} = 0$ we have that $*\tilde{\alpha}$ is proportional to $L^k\tilde{\alpha}$, therefore $\tilde{\alpha}$ is a representative of $[\alpha]$ satisfying $\delta\tilde{\alpha} = [\iota_G, d]\tilde{\alpha} = 0$.

Finally, if $[\alpha] \in H^{n+k}_{\text{hr}}(\mathfrak{g}, \omega)$ then there is $\beta \in \bigwedge^{n-k}_{\text{hr}}(\mathfrak{g}^*, \omega)$ such that $\alpha = L^k\beta$, so $H^{n+k}_{\text{hr}}(\mathfrak{g}, \omega) = L^k(H^{n-k}_{\text{hr}}(\mathfrak{g}, \omega))$. $\boxed{QED}$

Suppose that a simply connected Lie group G has a discrete subgroup Γ such that the quotient $M = \Gamma\backslash G$ is compact. Let us denote by $\mathfrak{g}$ the Lie algebra of G. Since any element in $\bigwedge^k(\mathfrak{g}^*)$ is identified to a left invariant form on G, it descends to the quotient M and there is a natural injection $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ which commutes with the differentials.

On the other hand, if the Lie algebra $\mathfrak{g}$ of G possesses a symplectic form ω then it descends to a symplectic form on M, which we shall also denote by ω. In this case the natural injection $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ also commutes with the symplectic stars, and so with the δ's. Therefore, we have a natural homomorphism $H^*_{\text{hr}}(\mathfrak{g}, \omega) \longrightarrow H^*_{\text{hr}}(M)$.

Proposition 4.3 *If the natural inclusion $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ induces an isomorphism $H^*(\mathfrak{g}) \cong H^*(M)$ in cohomology, then the inclusion $\bigwedge^*_{\text{hr}}(\mathfrak{g}^*, \omega) \hookrightarrow \Omega^*_{\text{hr}}(M)$ also induces an isomorphism $H^*_{\text{hr}}(\mathfrak{g}, \omega) \cong H^*_{\text{hr}}(M)$.*

Proof : Since the natural homomorphism $H^k(\mathfrak{g}) \longrightarrow H^k(M)$ commutes with L and it is an isomorphism, for each $k \leq n$ we have an isomorphism between $P_k(\mathfrak{g}, \omega)$ and the space $P_k(M) = \{[\alpha] \in H^k(M) \mid L^{n-k+1}[\alpha] = 0\}$ of primitive cohomology classes of degree k. Since $H^{n+k}_{\mathrm{hr}}(M) = L^k(H^{n-k}_{\mathrm{hr}}(M))$, from Lemma 4.2 it suffices to prove that $H^{n-k}_{\mathrm{hr}}(\mathfrak{g}, \omega) \cong H^{n-k}_{\mathrm{hr}}(M)$. But this follows easily by an inductive argument, taking into account Lemma 4.2 and the fact [21] that $H^{n-k}_{\mathrm{hr}}(M) = P_{n-k}(M) + L(H^{n-k-2}_{\mathrm{hr}}(M))$. Notice that for starting the induction, i.e. for $n - k = 0, 1, 2$, we have $H^{n-k}_{\mathrm{hr}}(M) = H^{n-k}(M) \cong H^{n-k}(\mathfrak{g}) = H^{n-k}_{\mathrm{hr}}(\mathfrak{g}, \omega)$. $\boxed{QED}$

Let $M = \Gamma\backslash G$ be a compact solvmanifold, that is, a compact quotient of a simply connected solvable Lie group G by a discrete subgroup Γ. Suppose in addition that the Lie algebra $\mathfrak{g}$ of G is completely solvable, i.e. $\mathrm{ad}_X \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ has only real eigenvalues for any $X \in \mathfrak{g}$. By Hattori theorem [9], which is a generalization to the completely solvable context of Nomizu theorem [18] for nilmanifolds, the natural inclusion $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ induces an isomorphism $H^*(\mathfrak{g}) \cong H^*(M)$.

Let ω be a symplectic form on $M = \Gamma\backslash G$. From the results above it is clear that the harmonic cohomology only depends on the cohomology class $[\omega]$ of the symplectic form. Since $H^2(M) \cong H^2(\mathfrak{g})$ we can suppose without loss of generality that ω is invariant, that is, it stems from a symplectic form on the Lie algebra $\mathfrak{g}$.

Corollary 4.4 *Let $M = \Gamma\backslash G$ be a compact solvmanifold endowed with a symplectic form ω. If the Lie algebra $\mathfrak{g}$ of G is completely solvable, then the natural injection $\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*, \omega) \hookrightarrow \Omega^*_{\mathrm{hr}}(M)$ induces an isomorphism $H^*_{\mathrm{hr}}(\mathfrak{g}, \omega) \cong H^*_{\mathrm{hr}}(M)$.*

In particular, the result holds for symplectic nilmanifolds, which has been already obtained in [21].

From Theorems 1.4 and 1.5 and their corresponding analogues for a Lie algebra endowed with a symplectic form, we have the following results.

Corollary 4.5 *Let $M = \Gamma\backslash G$ be a compact solvmanifold endowed with a symplectic form ω. Then, the $d\delta$-lemma holds on M up to degree s if and only if it holds on $\mathfrak{g}$ up to degree s, i.e.,*

$$\begin{aligned} &d(\textstyle\bigwedge^{k-1}(\mathfrak{g}^*)) \cap \ker\delta = \delta(\textstyle\bigwedge^{k+1}(\mathfrak{g}^*)) \cap \ker d = d\delta(\textstyle\bigwedge^{k}(\mathfrak{g}^*)), \qquad \text{for } k \leq s, \\ &d(\textstyle\bigwedge^{s}(\mathfrak{g}^*)) \cap \ker\delta = d\delta(\textstyle\bigwedge^{s+1}(\mathfrak{g}^*)). \end{aligned}$$

Corollary 4.6 *Let $M = \Gamma\backslash G$ be a compact solvmanifold endowed with a symplectic form ω. Then, the map i given in (2) is bijective for all $k \geq 2n - s$ if and only if the map $i \colon H^k_\delta(\mathfrak{g}, \omega) \longrightarrow H^k(\mathfrak{g})$ is bijective for all $k \geq 2n - s$, where $H^k_\delta(\mathfrak{g}, \omega)$ denotes the cohomology of $(\bigwedge^*_\delta(\mathfrak{g}^*, \omega) = \{\alpha \in \bigwedge^*(\mathfrak{g}^*) \mid \delta\alpha = 0\},\ d)$.*

Notice that Theorems 1.4 and 1.5 imply that if two symplectic forms ω and ω' are cohomologous then the $d\delta$-lemma holds up to degree s for ω if and only if it does for ω', and the map i given in (2) is bijective for all $k \geq 2n - s$ for ω if and only if it is so for ω'. Therefore, we can consider symplectic forms up to cohomology class.

Next we consider an arbitrary symplectic form on some examples of compact completely solvable manifolds, where we show explicit calculations.

Example 4.7 *The Kodaira-Thurston manifold.* Let G be the connected nilpotent Lie group of dimension 4 given by $G = H \times \mathbb{R}$, where H is the Heisenberg group, that is, the Lie group consisting of matrices of the form

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. If Γ' denotes the discrete subgroup of H consisting of matrices whose entries x, y and z are integer numbers, then the quotient space $KT = \Gamma\backslash G$, where $\Gamma = \Gamma' \times \mathbb{Z}$, is a compact manifold.

A global system of coordinates (x, y, z) for H is given by $x(g) = x$, $y(g) = y$, $z(g) = z$, and a standard calculation shows that a basis for the left invariant 1–forms on H consists of $\{dx, dy, dz - xdy\}$. Thus, if t denotes the standard coordinate for $\mathbb{R}$, then $\{\alpha = -dx, \beta = dy, \gamma = dt, \tau = dz - xdy\}$ is a basis of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G with Chevalley-Eilenberg differential given by

$$d\alpha = d\beta = d\gamma = 0, \quad d\tau = \alpha \wedge \beta.$$

So, the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ is given by

$$\begin{array}{rcl} H^0(\mathfrak{g}) & = & \langle 1 \rangle, \\ H^1(\mathfrak{g}) & = & \langle [\alpha], [\beta], [\gamma] \rangle, \\ H^2(\mathfrak{g}) & = & \langle [\alpha \wedge \gamma], [\alpha \wedge \tau], [\beta \wedge \gamma], [\beta \wedge \tau] \rangle, \\ H^3(\mathfrak{g}) & = & \langle [\alpha \wedge \beta \wedge \tau], [\alpha \wedge \gamma \wedge \tau], [\beta \wedge \gamma \wedge \tau] \rangle, \\ H^4(\mathfrak{g}) & = & \langle [\alpha \wedge \beta \wedge \gamma \wedge \tau] \rangle. \end{array}$$

For any element $\omega \in \bigwedge^2(\mathfrak{g}^*)$ satisfying $d\omega = 0$ there exists $a, b, c, e \in \mathbb{R}$ such that

$$[\omega] = a\,[\alpha \wedge \gamma] + b\,[\beta \wedge \gamma] + c\,[\alpha \wedge \tau] + e\,[\beta \wedge \tau].$$

Since $[\omega]^2 = 2(bc - ae)[\alpha \wedge \beta \wedge \gamma \wedge \tau]$, we conclude that $[\omega]^2 \neq 0$ if and only if $ae \neq bc$. Hence, up to cohomology class, we can consider that any symplectic form on $\mathfrak{g}$ is given by

$$\omega = a\,\alpha \wedge \gamma + b\,\beta \wedge \gamma + c\,\alpha \wedge \tau + e\,\beta \wedge \tau, \qquad ae - bc \neq 0. \tag{8}$$

Moreover, notice that for the new basis of $\mathfrak{g}^*$ given by

$$\alpha' = (ae - bc)(a\,\alpha + b\,\beta), \quad \beta' = \frac{1}{ae - bc}(c\,\alpha + e\,\beta), \quad \gamma' = \frac{1}{ae - bc}\,\gamma, \quad \tau' = (ae - bc)\tau,$$

the differential d expressed again as

$$d\alpha' = d\beta' = d\gamma' = 0, \quad d\tau' = \alpha' \wedge \beta'.$$

Now, with respect to this basis the symplectic form (8) is given by

$$\omega = \alpha' \wedge \gamma' + \beta' \wedge \tau'.$$

Therefore, we can suppose without loss of generality that $a = e = 1$ and $b = c = 0$ in (8).

Observe that $[\omega] \cup [\beta] = 0$ in $H^3(\mathfrak{g})$, and $\dim H^3_{\rm hr}(\mathfrak{g}, \omega) = 2 < 3 = \dim H^3(\mathfrak{g})$. It follows from Corollary 4.4 that for any symplectic form ω on KT, the compact symplectic manifold (KT, ω) is not 1–Lefschetz, and $H^k_{\rm hr}(KT, \omega) = H^k(KT)$ for $k \neq 3$, but $\dim H^3_{\rm hr}(KT, \omega) = 2 < 3 = b_3(KT)$. Notice that any non-toral compact symplectic nilmanifold $(M = \Gamma\backslash G, \omega)$ is 0–Lefschetz but not 1–Lefschetz [1].

We study next the $d\delta$-lemma for any symplectic form ω on the Kodaira-Thurston manifold. By Corollary 4.5, the $d\delta$-lemma is satisfied up to degree $s = 0$ if and only if it is satisfied at the level of the Lie algebra $\mathfrak{g}$. Let us denote by $\{X, Y, Z, T\}$ the basis of $\mathfrak{g}$ dual to $\{\alpha, \beta, \gamma, \tau\}$, and let ω be a symplectic form on $\mathfrak{g}$ given by (8) with $a = e = 1$ and $b = c = 0$. Then, the isomorphism $\natural\colon \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given by

$$\natural(X) = \gamma, \quad \natural(Y) = \tau, \quad \natural(Z) = -\alpha, \quad \natural(T) = -\beta.$$

Therefore,
$$G = -\natural^{-1}(\omega) = -X \wedge Z - Y \wedge T.$$

In degree 1, we must determine the spaces $\delta(\bigwedge^2(\mathfrak{g}^*)) \cap \ker d$, $d(\bigwedge^0(\mathfrak{g}^*)) \cap \ker \delta$ and $d\delta(\bigwedge^1(\mathfrak{g}^*))$. Notice that $\delta(\bigwedge^1(\mathfrak{g}^*)) \subset \bigwedge^0(\mathfrak{g}^*) = \mathbb{R}$ and $d(\bigwedge^0(\mathfrak{g}^*)) = \{0\}$. Using that $\delta\mu = i_G(d\mu)$ for any $\mu \in \bigwedge^2(\mathfrak{g}^*)$, an easy calculation shows that $\delta(\bigwedge^2(\mathfrak{g}^*)) = \langle \beta \rangle$, in fact $\beta = \delta(-\gamma \wedge \tau)$. Since $\ker d = \langle \alpha, \beta, \gamma \rangle$, we have
$$\delta(\bigwedge^2(\mathfrak{g}^*)) \cap \ker d = \langle \beta \rangle \neq \{0\} = d\delta(\bigwedge^1(\mathfrak{g}^*)),$$
and the $d\delta$-lemma is not satisfied in degree 1. Moreover, $d(\bigwedge^1(\mathfrak{g}^*)) \cap \ker \delta = \langle \alpha \wedge \beta \rangle \neq \{0\} = d\delta(\bigwedge^2(\mathfrak{g}^*))$.

Applying the symplectic star operator, we get that the element $\alpha \wedge \beta \wedge \gamma = *\beta \in d(\bigwedge^2(\mathfrak{g}^*)) \cap \ker \delta$, but it does not belong to the space $d\delta(\bigwedge^3(\mathfrak{g}^*)) = \{0\}$.

Therefore, for any symplectic form ω on KT the $d\delta$–lemma is satisfied only up to degree 0, according to Theorem 1.4.

Notice that in general for any symplectic form on a nilpotent Lie algebra the map L^{n-1} is never injective [1]. From Theorem 1.4 it follows that $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ on $\bigwedge^1(\mathfrak{g}^*)$ and $\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta$ on $\bigwedge^{2n-1}(\mathfrak{g}^*)$, in fact these spaces are all zero, but either $\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta$ fails on $\bigwedge^1(\mathfrak{g}^*)$ or $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ fails on $\bigwedge^2(\mathfrak{g}^*)$. By duality, $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ fails on $\bigwedge^{2n-1}(\mathfrak{g}^*)$, or $\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta$ fails on $\bigwedge^{2n-2}(\mathfrak{g}^*)$.

Finally, we study the cohomology H^*_δ. At the level of $\mathfrak{g}$, the cohomology groups $H^k_\delta(\mathfrak{g}, \omega)$ are:
$$\begin{aligned}
H^0_\delta(\mathfrak{g}, \omega) &= \langle 1 \rangle,\\
H^1_\delta(\mathfrak{g}, \omega) &= \langle [\alpha], [\beta], [\gamma] \rangle,\\
H^2_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \gamma], [\alpha \wedge \tau], [\beta \wedge \gamma], [\beta \wedge \tau] \rangle,\\
H^3_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \beta \wedge \gamma], [\alpha \wedge \gamma \wedge \tau], [\beta \wedge \gamma \wedge \tau] \rangle,\\
H^4_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \beta \wedge \gamma \wedge \tau] \rangle.
\end{aligned}$$
Therefore, $i \colon H^k_\delta(\mathfrak{g}, \omega) \longrightarrow H^k(\mathfrak{g})$ is bijective for all $k \neq 3$, because $i([\alpha \wedge \beta \wedge \gamma]) = 0$ in $H^3(\mathfrak{g})$, in fact $\alpha \wedge \beta \wedge \gamma = d(-\gamma \wedge \tau)$. ¿From Corollary 4.6 we have that for any symplectic form ω on KT the map $i \colon H^k_\delta(KT, \omega) \longrightarrow H^k(KT)$ is bijective for $k = 4$, but not for $k = 3$, according to Theorem 1.5.

Example 4.8 *A six-dimensional solvmanifold.* Let G be the connected completely solvable Lie group of dimension 6 consisting of matrices of the form
$$g = \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1\\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2\\ 0 & 0 & e^t & 0 & 0 & z_1\\ 0 & 0 & 0 & e^{-t} & 0 & z_2\\ 0 & 0 & 0 & 0 & 1 & t\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
where $t, x, y_i, z_i \in \mathbb{R}$ $(i = 1, 2)$. The Lie group G has a discrete subgroup Γ such that the quotient space $M = \Gamma \backslash G$ is compact [5].

A global system of coordinates $(t, x, y_1, y_2, z_1, z_2)$ for G is defined by $t(g) = t$, $x(g) = x$, $y_i(g) = y_i$, $z_i(g) = z_i$, and a standard calculation shows that a basis for the left invariant 1–forms on G consists of
$$\{\alpha = dt,\ \beta = dx,\ \gamma_1 = e^{-t}dy_1 - xe^{-t}dz_1,\ \gamma_2 = e^t dy_2 - xe^t dz_2,\ \tau_1 = e^{-t}dz_1,\ \tau_2 = e^t dz_2\}.$$

Hence, $\{\alpha, \beta, \gamma_1, \gamma_2, \tau_1, \tau_2\}$ is a basis of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G with Chevalley-Eilenberg differential given by

$$d\alpha = d\beta = 0, \quad d\gamma_1 = -\alpha\wedge\gamma_1 - \beta\wedge\tau_1, \quad d\gamma_2 = \alpha\wedge\gamma_2 - \beta\wedge\tau_2, \quad d\tau_1 = -\alpha\wedge\tau_1, \quad d\tau_2 = \alpha\wedge\tau_2.$$

Now, a direct calculation shows that the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ is given by

$$\begin{aligned}
H^0(\mathfrak{g}) &= \langle 1\rangle,\\
H^1(\mathfrak{g}) &= \langle [\alpha], [\beta]\rangle,\\
H^2(\mathfrak{g}) &= \langle [\alpha\wedge\beta], [\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1], [\tau_1\wedge\tau_2]\rangle,\\
H^3(\mathfrak{g}) &= \langle [\alpha\wedge(\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1)], [\alpha\wedge\tau_1\wedge\tau_2], [\beta\wedge\gamma_1\wedge\gamma_2], [\beta\wedge(\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1)]\rangle,\\
H^4(\mathfrak{g}) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2], [\alpha\wedge\beta\wedge(\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1)], [\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle,\\
H^5(\mathfrak{g}) &= \langle [\alpha\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2], [\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle,\\
H^6(\mathfrak{g}) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle.
\end{aligned}$$

For any element $\omega \in \bigwedge^2(\mathfrak{g}^*)$ satisfying $d\omega = 0$ there exists $a, b, c \in \mathbb{R}$ such that

$$[\omega] = a\,[\alpha\wedge\beta] + b\,[\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1] + c\,[\tau_1\wedge\tau_2].$$

Since $[\omega]^3 = 6ab^2[\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]$, we conclude that $[\omega]^3 \neq 0$ if and only if $ab \neq 0$.

Thus, up to cohomology class, we can consider that any symplectic form on $\mathfrak{g}$ is given by

$$\omega = a\,\alpha\wedge\beta + b\,\gamma_1\wedge\tau_2 + b\,\gamma_2\wedge\tau_1 + c\,\tau_1\wedge\tau_2, \qquad a, b \neq 0. \tag{9}$$

Let us consider the new basis of $\mathfrak{g}^*$ given by

$$\alpha' = \alpha,\ \beta' = a\,\beta,\ \gamma_1' = \sqrt{\frac{a}{b}}\,\left(b\gamma_1 + \frac{c}{2}\,\tau_1\right),\ \gamma_2' = \sqrt{\frac{a}{b}}\,\left(b\gamma_2 - \frac{c}{2}\,\tau_2\right),\ \tau_1' = \sqrt{\frac{b}{a}}\,\tau_1,\ \tau_2' = \sqrt{\frac{b}{a}}\,\tau_2,$$

if $ab > 0$, or

$$\alpha' = -\alpha,\ \beta' = -a\,\beta,\ \gamma_1' = \sqrt{-\frac{a}{b}}\,\left(b\gamma_2 - \frac{c}{2}\,\tau_2\right),$$

$$\gamma_2' = \sqrt{-\frac{a}{b}}\,\left(b\gamma_1 + \frac{c}{2}\,\tau_1\right),\ \tau_1' = \sqrt{-\frac{b}{a}}\,\tau_2,\ \tau_2' = \sqrt{-\frac{b}{a}}\,\tau_1,$$

if $ab < 0$. The differential d also expressed as

$$d\alpha' = d\beta' = 0, \quad d\gamma_1' = -\alpha'\wedge\gamma_1' - \beta'\wedge\tau_1', \quad d\gamma_2' = \alpha'\wedge\gamma_2' - \beta'\wedge\tau_2', \quad d\tau_1' = -\alpha'\wedge\tau_1', \quad d\tau_2' = \alpha'\wedge\tau_2'.$$

With respect to this basis the symplectic form (9) is given by

$$\omega = \alpha'\wedge\beta' + \gamma_1'\wedge\tau_2' + \gamma_2'\wedge\tau_1',$$

so we can suppose without loss of generality that $a = b = 1$ and $c = 0$ in (9).

Observe that $[\omega] \cup [\tau_1\wedge\tau_2] = 0$ in $H^4(\mathfrak{g})$, but a simple computation shows that the product by $[\omega]^2$ is an isomorphism between $H^1(\mathfrak{g})$ and $H^5(\mathfrak{g})$. Moreover, $\dim H^4_{\mathrm{hr}}(\mathfrak{g}, \omega) = 2 < 3 = \dim H^4(\mathfrak{g})$. Therefore, for any symplectic form ω on M, the compact symplectic manifold (M, ω) is 1–Lefschetz, but not 2–Lefschetz, and Corollary 4.4 implies that $\dim H^k_{\mathrm{hr}}(M, \omega) = b_k(M)$ for $k \neq 4$, but $\dim H^4_{\mathrm{hr}}(M, \omega) = 2 < 3 = b_4(M)$.

Next we study the $d\delta$-lemma for any symplectic form on the compact solvmanifold M. Corollary 4.5 implies that the $d\delta$-lemma is satisfied up to degree 1 on M if and only if it is satisfied on $\mathfrak{g}$. Let us denote by $\{X, Y, Z_1, Z_2, T_1, T_2\}$ the basis of $\mathfrak{g}$ dual to $\{\alpha, \beta, \gamma_1, \gamma_2, \tau_1, \tau_2\}$, and let ω be a symplectic form on $\mathfrak{g}$ given by (9) with $a = 1$, $b = 1$ and $c = 0$. Then, the isomorphism $\natural\colon \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given by

$$\natural(X) = \beta, \quad \natural(Y) = -\alpha, \quad \natural(Z_1) = \tau_2, \quad \natural(Z_2) = \tau_1, \quad \natural(T_1) = -\gamma_2, \quad \natural(T_2) = -\gamma_1.$$

Therefore, $G = -\natural^{-1}(\omega)$ is given by

$$G = -X \wedge Y - Z_1 \wedge T_2 - Z_2 \wedge T_1.$$

In degree 1 we must consider the spaces $\delta(\bigwedge^2(\mathfrak{g}^*)) \cap \ker d$, $d(\bigwedge^0(\mathfrak{g}^*)) \cap \ker \delta$ and $d\delta(\bigwedge^1(\mathfrak{g}^*))$. Since $\delta(\bigwedge^1(\mathfrak{g}^*)) \subset \bigwedge^0(\mathfrak{g}^*) = \mathbb{R}$, the $d\delta$-lemma is satisfied in degree 1 if and only if

$$\delta(\textstyle\bigwedge^2(\mathfrak{g}^*)) \cap \ker d = \{0\}.$$

Using that $\delta\mu = i_G(d\mu)$ for any $\mu \in \bigwedge^2(\mathfrak{g}^*)$, a direct calculation shows that the space $\delta(\bigwedge^2(\mathfrak{g}^*))$ is generated by γ_1, γ_2, τ_1 and τ_2. Since $\ker d = \langle \alpha, \beta \rangle$, the $d\delta$-lemma holds in degree 1.

In degree 2 we must compare the spaces $\delta(\bigwedge^3(\mathfrak{g}^*)) \cap \ker d$, $d(\bigwedge^1(\mathfrak{g}^*)) \cap \ker \delta$ and $d\delta(\bigwedge^2(\mathfrak{g}^*))$. It is easy to check that $d(\bigwedge^1(\mathfrak{g}^*)) \subset \ker\{\delta\colon \bigwedge^2(\mathfrak{g}^*) \longrightarrow \bigwedge^1(\mathfrak{g}^*)\}$. Therefore,

$$d(\textstyle\bigwedge^1(\mathfrak{g}^*)) \cap \ker \delta = d(\bigwedge^1(\mathfrak{g}^*)) = \langle d\gamma_1, d\gamma_2, d\tau_1, d\tau_2 \rangle = d\delta(\bigwedge^2(\mathfrak{g}^*)),$$

so this space is generated by $\alpha \wedge \gamma_1 + \beta \wedge \tau_1$, $\alpha \wedge \gamma_2 - \beta \wedge \tau_2$, $\alpha \wedge \tau_1$ and $\alpha \wedge \tau_2$.

However, a long but direct calculation shows that

$$\delta(\textstyle\bigwedge^3(\mathfrak{g}^*)) \cap \ker d = \langle d\gamma_1, d\gamma_2, d\tau_1, d\tau_2, \tau_1 \wedge \tau_2 \rangle \;\not\subset\; d(\bigwedge^1(\mathfrak{g}^*)) \cap \ker \delta.$$

In fact, notice that

$$\begin{aligned} \delta(\alpha \wedge \gamma_1 \wedge \tau_2) &= i_G d(\alpha \wedge \gamma_1 \wedge \tau_2) - d i_G(\alpha \wedge \gamma_1 \wedge \tau_2) \\ &= i_G(\alpha \wedge \beta \wedge \tau_1 \wedge \tau_2) + d(\alpha) \\ &= -\tau_1 \wedge \tau_2, \end{aligned}$$

and $d(\tau_1 \wedge \tau_2) = 0$. Thus, the $d\delta$-lemma is satisfied up to degree 1, but it does not hold up to degree 2. Therefore, for any symplectic form ω on M the $d\delta$–lemma is satisfied only up to degree 1, according to Theorem 1.4.

Notice that the element $\alpha \wedge \beta \wedge \tau_1 \wedge \tau_2 = *(-\tau_1 \wedge \tau_2) \in d(\bigwedge^3(\mathfrak{g}^*)) \cap \ker \delta$ does not belong to the space $d\delta(\bigwedge^4(\mathfrak{g}^*))$.

Finally, the cohomology groups $H^k_\delta(\mathfrak{g}, \omega)$ are given by:

$$\begin{aligned} H^0_\delta(\mathfrak{g}, \omega) &= \langle 1 \rangle, \\ H^1_\delta(\mathfrak{g}, \omega) &= \langle [\alpha], [\beta] \rangle, \\ H^2_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \beta], [\gamma_1 \wedge \tau_2 + \gamma_2 \wedge \tau_1], [\tau_1 \wedge \tau_2] \rangle, \\ H^3_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge (\gamma_1 \wedge \tau_2 + \gamma_2 \wedge \tau_1)], [\alpha \wedge \tau_1 \wedge \tau_2], [\beta \wedge \gamma_1 \wedge \gamma_2], [\beta \wedge (\gamma_1 \wedge \tau_2 + \gamma_2 \wedge \tau_1)] \rangle, \\ H^4_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \beta \wedge (\gamma_1 \wedge \tau_2 + \gamma_2 \wedge \tau_1)], [\alpha \wedge \beta \wedge \tau_1 \wedge \tau_2], [\gamma_1 \wedge \gamma_2 \wedge \tau_1 \wedge \tau_2] \rangle, \\ H^5_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \gamma_1 \wedge \gamma_2 \wedge \tau_1 \wedge \tau_2], [\beta \wedge \gamma_1 \wedge \gamma_2 \wedge \tau_1 \wedge \tau_2] \rangle, \\ H^6_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \beta \wedge \gamma_1 \wedge \gamma_2 \wedge \tau_1 \wedge \tau_2] \rangle. \end{aligned}$$

Thus, the map $i\colon H^k_\delta(\mathfrak{g},\omega) \longrightarrow H^k(\mathfrak{g})$ is bijective for all $k \neq 4$. In fact, since $d(\alpha\wedge\gamma_1\wedge\tau_2) = \alpha\wedge\beta\wedge\tau_1\wedge\tau_2$ we have that $i([\alpha\wedge\beta\wedge\tau_1\wedge\tau_2]) = 0$ in $H^4(\mathfrak{g})$. By Corollary 4.6 we conclude that for any symplectic form ω on M the map $i\colon H^k_\delta(M,\omega) \longrightarrow H^k(M)$ is bijective for $k = 5, 6$, but not for $k = 4$, according to Theorem 1.5.

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